Designing Materials for Mechanical Invisibility Cloaks

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Abstract Transformation elastodynamics, the solid mechanical counterpart of transformation optics, is an approach to re-routing of mechanical, potentially harmful transient waves and vibrations, to protect structures or substructures from harm. A large-scale example would be to try to re-route seismic waves, whether from ground explosions or earthquakes, by arranging the properties of the material beneath and around some sensitive infrastructure so as to mimic the surrounding soil without any infrastructure. A less ambitious (and considerably more realistic) application would be to re-route elastic vibrations around the clamping points of panels in a vehicle, so as to minimize the noise from vibrating panels. Just as for transformation optics, the approach utilizes the concept of form-invariance of the equations of motion under diffeomorphisms to give recipes of how graded materials can mimic homogeneous and isotropic bodies, and cloak the presence of structures within the transformational cloak.

We have studied the use of several types of graded materials for cloaking, and in the present paper we describe how graded micropolar materials may be used to cloak against Rayleigh waves. We have implemented recipes for the graded properties of a micropolar cloak from transformation elastodynamics into a modified version of the Structural Mechanics module of COMSOL Multiphysics™. In numerical experiments we consider how a modeled, partially buried 'pipeline' may be protected from an incident transient Rayleigh wave by re-routing the wave under the pipeline.

1. Introduction

Less than a decade ago, it was discovered that invisibility cloaking devices could be more than mere fiction. Originally for a type of 'X-ray' technique, electric surface impedance tomography, where a voltage distribution is applied to the surface of a body, and resulting electric currents through the surface are measured, with a view to reconstructing the conductivity inside the object. Theoretically, the interior of the object could, under certain circumstances, hide (or 'cloak') an embedded body in such a manner that even the fact that something was hidden would be undetectable. Soon similar phenomena were shown to be possible also for electromagnetic waves.

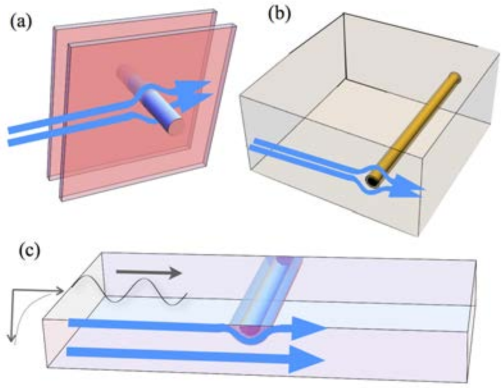

Figure 1: *Some geometries where mechanical cloaking might be desirable: a) Clamping point between panels. b) Buried infrastructure. c) Surface-breaking structure.*

Also in solid mechanics, there is considerable interest in achieving 'invisibility,' however not primarily for hiding objects from sight. Important contributions to elastodynamic transformational cloaking have been given by, among others [2][7][5][3][8]. Recently, an approach to partial cloaking using fiber composites has been explored, see *e.g* [9]. The applications in mechanics include protection of structures and parts of structures from potentially harmful transient waves and steady state vibrations, see Figure 1 for some generic examples.

Another, similar type of situation would be ground waves from trains or other vehicles in rapid transit. These waves could possibly be redirected so as to protect buildings situated too close to the tracks or roads.

A suggested larger scale application, protection against seismic waves from earthquakes, could be achieved by using cloaking to re-route the waves around sensitive infras-

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tructure, *cf.* Figure 2. While the cost for such a protection would be astronomical, it should be compared to the likewise astronomical cost in case of a large-scale collapse of *e.g.* a nuclear power plant due to seismic waves.

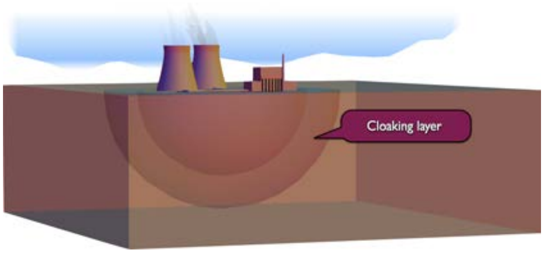

Figure 2: *A potential, very(!) ambitious, and astronomically expensive application of mechanical cloaking.*

At smaller scales, an application could be to re-direct elastic waves around e.g. clamping points for panels in vehicles or other structures, thereby achieving some noise control by entirely passive means. On the very small scale, protection of sensitive electronic components from vibrations might be accomplished by surrounding the components by a suitable mechanical cloak.

The construction of mechanical cloaks requires fine-tuning of the elastic properties of the cloaking material, so-called meta-materials [1]. Multiscale mechanics is the topic of connecting different length scales in mechanics. For solid materials, the macroscale response is defined through suitable microscale models that resolve the actual topology on the lower scale. The straight-forward question in multiscale mechanics (or micromechanics) is what the macroscale response will be for a given structure on the microscale. One classical example is homogenization of elastic properties for micro-heterogeneous materials [6]. However, in an inverse setting, multiscale mechanics (or homogenization [4]) can also be used to design the microstructure such that a specific response on the macroscale is obtained.

We describe some results on simulations of cloaking in solid mechanics. COMSOL Multiphysics™ has been used to simulate different material responses. In order to apply the anti-symmetric elasticity matrix properties that was needed, a modification of the weak formulation in the software was made. In the next step, simulations of homogenization are required of micro-heterogeneous materials to find meta-materials suitable for producing the required macro-properties for cloaking.

2. Governing Equations

A general (hemitropic) micropolar medium satisfies the constitutive equations

$$\begin{aligned}\boldsymbol{\sigma} &= \mathsf{C} : (\nabla \otimes \mathbf{u} - \boldsymbol{\epsilon} \cdot \boldsymbol{\phi}) + \mathsf{B} : (\nabla \otimes \boldsymbol{\phi}) \\ \boldsymbol{\mu} &= \mathsf{B}^{\mathsf{T}} : (\nabla \otimes \mathbf{u} - \boldsymbol{\epsilon} \cdot \boldsymbol{\phi}) + \mathsf{A} : (\nabla \otimes \boldsymbol{\phi})\end{aligned}$$

and the equations of motion

$$\begin{aligned}\nabla \cdot \boldsymbol{\sigma}^{\mathsf{T}} &= \rho\, \ddot{\mathbf{u}} \\ \nabla \cdot \boldsymbol{\mu}^{\mathsf{T}} + \boldsymbol{\epsilon} : \boldsymbol{\sigma} &= \boldsymbol{\jmath} \cdot \ddot{\boldsymbol{\phi}}\end{aligned}$$

The totally anti-symmetric third order tensor in three diemensions may be defined as $\boldsymbol{\epsilon} = \mathbf{I} \times \mathbf{I}$, where $\mathbf{I}$ is the second order unit tensor. The double contraction is defined so that $\mathbf{X} : \mathbf{Y} = X_{ij} Y_{ij}$ for $\mathbf{X} = X_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathbf{Y} = Y_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$. Similarly $\mathsf{K} : \mathbf{Y} = K_{ijk\ell} Y_{k\ell} \mathbf{e}_i \otimes \mathbf{e}_j$ for $\mathsf{K} = K_{ijk\ell}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell$. C, A are major symmetric tensors of order 4. The symmetry is under transpose of the first and last index *pairs*. We use boldface superscript T to denote this transposition, and ordinary sans serif T for transpose of second-order tensors. The major symmetry requirement can thus be stated as

$$\begin{aligned}\mathsf{A}^{\mathsf{T}} &= A_{k\ell ij} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell \\ &= A_{ijk\ell} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell = \mathsf{A}\end{aligned}$$

and similarly for C. Note that neither of these tensors need satisfy the minor symmetries, whereby *e.g.* $C_{ijkl} \neq C_{ijlk}$ in general. B is a tensor of order 4, with no assumptions on symmetry.

We now assume three things. First, that the material is centro-symmetric so that $\mathsf{B} = \mathbf{0}$. Second, that time-harmonic conditions prevail, with time factor $\exp(-\mathrm{i}\omega t)$. And third, that the curvature stiffness is much higher than the stiffness with respect to strains. Then the tensor A in some suitable sense becomes very large, while the micro-moment tensor $\boldsymbol{\mu}$ remains finite.

To be slightly more specific regarding the last point, let's say that we *e.g.* have some scalar parameter a that we let tend to positive infinity, and that for some $\epsilon > 0$

$$\mathsf{A} = a\, \mathbf{M} \boxtimes \mathbf{M} + \mathcal{O}[a^{1-\epsilon}] \quad \text{as } a \to +\infty$$

where $\mathbf{M} = M_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ is some second order tensor. (We also assume that the matrix formed from its coefficients M_{ij} is invertible.)

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Here the boxed multiplication of two second order tensors is defined as

$$\mathbf{X} \boxtimes \mathbf{Y} = X_{ik} Y_{j\ell}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell$$

Then, if $\boldsymbol{\mu}$ is to remain finite, we must have that $\nabla \otimes \boldsymbol{\phi} \to \mathbf{0}$ as $a \to +\infty$, and $\boldsymbol{\phi} \to$ constant throughout the body. If the boundary condition is $\boldsymbol{\phi}\big|_{\partial\Omega} = \mathbf{0}$, and the limit of $\boldsymbol{\phi}$ is uniform, then in the limit $\boldsymbol{\phi}$ must vanish throughout Ω.

Under these three assumptions, the set of four equations implies

$$\boldsymbol{\sigma} = \mathbf{C} : (\nabla \otimes \mathbf{u}) \tag{1}$$

$$\nabla \bullet \boldsymbol{\sigma}^{\mathsf{T}} + \rho\, \omega^2\, \mathbf{u} = \mathbf{0} \quad \text{in } \Omega, \tag{2}$$

This means that, given suitable boundary conditions on the displacement, stress vector and the micro-moment tensor, the problem gives a boundary value problem for the displament. The field $\boldsymbol{\mu}$ may subsequently be retrieved. Eq. (1) and Eq. (2), must be supplemented by a boundary condition

$$\widehat{\mathbf{n}} \bullet \boldsymbol{\sigma}^{\mathsf{T}}\big|_{\partial\Omega} = \mathbf{h} \tag{3}$$

where $\mathbf{h}$ is a prescribed vector field defined on $\partial\Omega$, and $\widehat{\mathbf{n}}$ is the outward-pointing unit normal.

3. Cloaking Transformation

As found in a special case in [2], and more generally in [8], a body consisting of this kind of micropolar material can mimic a homogeneous isotropic elastic body in the following sense: If the body inside Ω is micropolar with a stiffness and density that varies in a suitable manner, the Traction-to-Displacement (TtD) map, that maps $\mathbf{h} \mapsto \mathbf{u}\big|_{\partial\Omega}$ for the micropolar solid, may be identically the same as that of a homogeneous, isotropic solid occupying the same region Ω.

Consider a diffeomorphism $\boldsymbol{\psi} : \Omega \to \Omega$ such that the limit of $\boldsymbol{\psi}$ on $\partial\Omega$ is the identity map. Let $\mathbf{c} = c_{ijk\ell}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell$ denote the elasticity tensor a homogeneous isotropic elastic material, and let ρ_0 be its constant mass density, and put

$$C_{ijk\ell}(\mathbf{X}) = J(\mathbf{X})\, \frac{\partial X^j}{\partial x^p}\, c_{ipkq}\, \frac{\partial X^\ell}{\partial x^q} \tag{4}$$

$$\rho(\mathbf{X}) = J(\mathbf{X}) \rho_0$$

where

$$\mathbf{x} = \boldsymbol{\psi}(\mathbf{X}), \quad J(\mathbf{X}) = \det\!\left(\frac{\partial \boldsymbol{\psi}(\mathbf{X})}{\partial \mathbf{X}}\right).$$

Note that the elasticity tensor of the micropolar material, with components given by Eq. (4), does indeed satisfy the major symmetry condition. Then we may consider the x^j as cartesian coordinates in the homogeneous isotropic solid, and the X^j as cartesian coordinates in the inhomogeneous, anisotropic micropolar material, each occupying Ω. If the displacement field in the homogeneous material is $\mathbf{u}(\mathbf{x})$, satisfying

$$\frac{\partial}{\partial x^j}\left(c_{ijpq} \frac{\partial u_p(\mathbf{x})}{\partial x^q} \right) + \rho_0\, \omega^2 u_i(\mathbf{x}) = 0$$

then $\mathbf{U}(\mathbf{X}) = \mathbf{u}(\boldsymbol{\psi}^{-1}(\mathbf{X}))$ satisfies

$$\frac{\partial}{\partial X^j}\left(C_{ijpq}(\mathbf{X}) \frac{\partial U_p(\mathbf{X})}{\partial X^q} \right) + \rho(\mathbf{X})\, \omega^2 U_i(\mathbf{X}) = 0$$

Even without special assumptions on the normal derivative of $\boldsymbol{\psi}$ on $\partial\Omega$, it may be verified that both tractions and displacements on the boundary $\partial\Omega$ coincide in the two cases. For any surface excitation of the micropolar body, the response at the boundary is the same as for a homogeneous body. The TtD maps of the two bodies are identical.

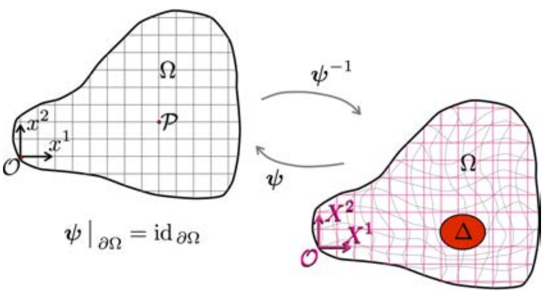

Figure 3: *'Blowing up' a point to make a hiding-place.*

By subtracting a point $\mathcal{P}$ in the interior from Ω, and choosing a diffeomorphism such that $\boldsymbol{\psi}^{-1}$ maps $\Omega\backslash\{\mathcal{P}\}$ to $\Omega\backslash\Delta$ where Δ is some open (simply connected *etc.*) subset of Ω, we may 'hide' an object within the micropolar solid by putting it inside Δ. The hidden object will in essence be invisible to any elastic waves impinging on the outer boundary $\partial\Omega$.

4. Rayleigh Wave Impinging on a Surface-Breaking Cylinder

Consider now the geometry indicated in Figure 4, where a pipeline is partially buried in the ground.

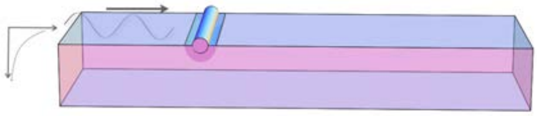

Figure 4: *A light pipeline, partially buried in a homogeneous and isotropic ground.*

We consider two types of support for the pipeline. First the layer between the soil and the pipeline is just made of the same homogeneous and isotropic material as the rest of the ground, and the pipeline is in 'welded' contact with the support. In the second case we consider the intermediate layer to be made of a micropolar material which is graded so that it cloaks the pipeline. The diffeomorphism chosen only affects the radial coordinate r, measured from the rotational symmetry axis of the pipeline. We put $\boldsymbol{\psi}(\mathbf{r}) = \phi(r)\mathbf{e}_r$, with $\phi(r)$ being an invertible map of $r_0 < r < r_1$ onto $0 < r < r_1$, where r_0 is the inner radius and r_1 the outer radius of the cloaking layer. Since we demand $\phi(r_1) = r_1$, the transformation (in 2D) must allow

$$\int_{r_0}^{r_1} \rho(r) r^2 dr = \frac{1}{2}\rho_0 r_1^2$$

which constitutes a necessary condition on the density function. (Incidentally, this guarantees that the total mass of the cloak is finite, and in fact equal to the mass of the homogeneous half cylinder it mimics.) There are of course an infinity of transformations of this type, and we chose one of these, namely

$$\phi(r) = r_1 \sqrt{\frac{r^2 - r_0^2}{r_1^2 - r_0^2}}$$

In addition to introducing the cloak, we decouple the pipeline further by allowing it to slide without friction on the cloaking layer.

In the simulations, the combined effect of the sliding BC and the cloak is to decrease the amplitude of the movement of the pipeline induced by the incident Rayleigh wave by at least five orders of magnitude.

In Figure 5 we see two snapshots as the (transient) Rayleigh wave impinges on the pipeline from the left in the picture.

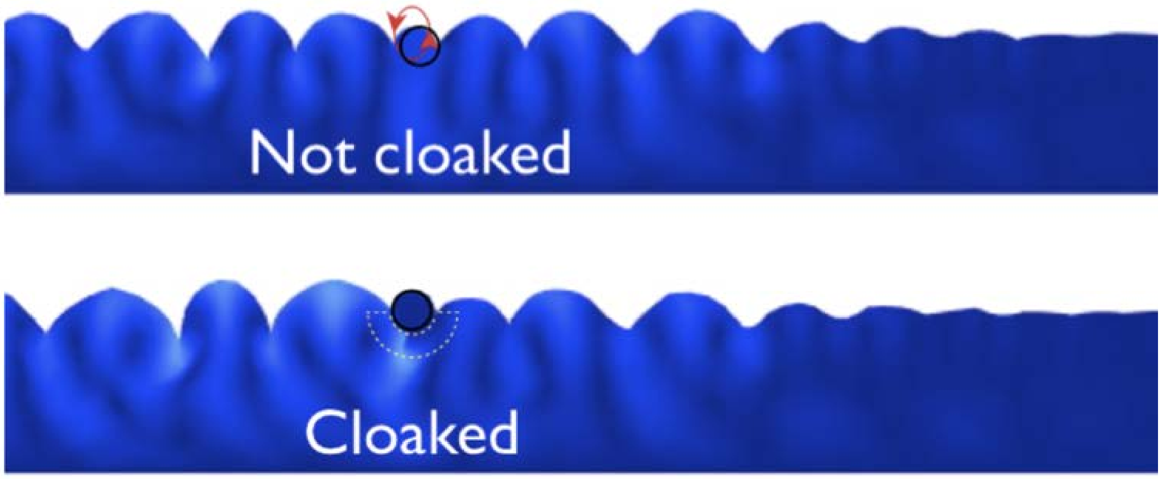

Figure 5: *Snapshots of transient Rayleigh wave incident upon an un-cloaked, welded, and a cloaked, sliding, pipeline, respectively. (Note that the scale of the displacement is immensely exaggerated, since we are actually in the linear regime.) The approximate path followed by the pipeline in the uncloaked case is indicated by an ellipse.*

In both cases the cylindrical pipeline rotates on a roughly elliptical path, counterclockwise, but the amplitude in the cloaked, sliding, case is so small that it cannot be discerned at this scale. From an animation of the movement, we find [2] that when the Rayleigh wave enters the cloaking layer, it essentially 'dives' under the pipeline and resurfaces behind the pipeline. In this way the cloak protects the pipeline from potential damage.

5. Implementation and Numerics

5.1 Global Definitions

Table 1: *Parameters*

Name	Expression
r0	0.2 [m]
r1	0.4 [m]
rho0	1 [kg/m^3]
mu0	1 [Pa]
lambda0	2.3 [Pa]
b	1 [N]
nu	lambda0/(2*(lambda0+mu0))
omega	6[rad/s]
ks	omega/(sqrt(mu0/rho0))
As	1[m^2]

To implement our model into COMSOL Multiphysics™, a number of parameters had to be defined, as indicated in Table 1. With these values all other important functions and parameters can be constructed and used in the software. The radial values, `r0` and `r1` describe the inner and outer radii of the cloaking region. `ks` and `As` are parameters used for the Rayleigh wave construction in the software while the rest are material parameters that describe the material characteristics.

5.2 Modifications to COMSOL Multiphysics™

In our model, the elasticity matrix is not symmetric as in the standard COMSOL Multiphysics™ Structural Mechanics module. The elasticity matrix in COMSOL Multiphysics™ is represented as in Figure 6.

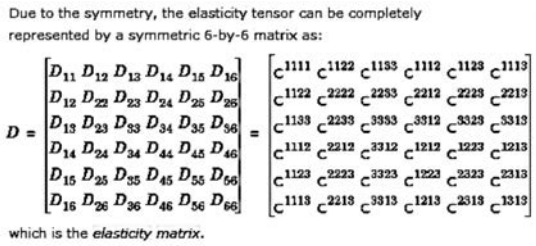

Figure 6: *Elasticity matrix in COMSOL Multiphysics™ Structural Mechanics module*

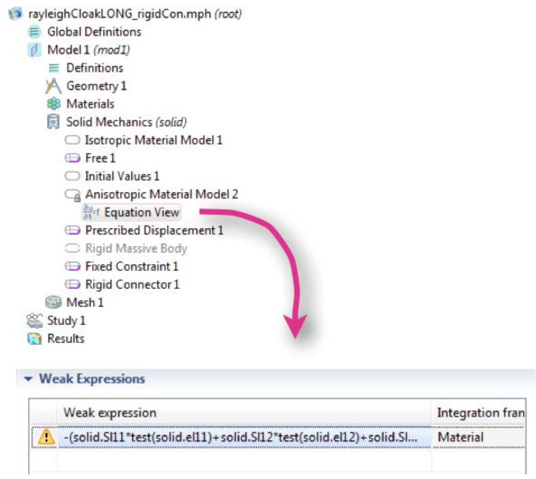

Figure 7: *Modifying module in COMSOL Multiphysics™*

The relevant module is based on weak form formulation, and may be modified to suit the problem on hand. In order to implement the non-symmetric 12 and 21 components, the weak formulation was modified in the Linear Elastic Material model, see Figure 7.

Table 2: *Modified stiffnesses.*

Name	Expression
C1111	((lambda0+2*mu0)*fi(sys2.r))/(sys2.r*fip(sys2.r))
C2222	((lambda0+2*mu0)*sys2.r*fip(sys2.r))/fi(sys2.r)
C1122	lambda0
C2211	lambda0
C1221	mu0
C2112	mu0
C1212	(mu0*fi(sys2.r))/(sys2.r*fip(sys2.r))
C2121	(mu0*sys2.r*fip(sys2.r))/fi(sys2.r)

The non-zero components of the modified elasticity matrix that were used in the model for the anisotropic cloaking material is indicated in Table 2.

5.3 Geometry

The geometry that was used needed to be elongated, in order to simulate uni-directional propagation of Rayleigh waves. In order to even better model the infinite medium around the model, perfectly matched layers (PML) were implemented.

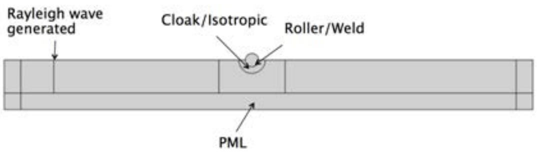

Figure 8: *The geometry*

Two different cases were simulated, one with a sliding, a.k.a. roller, boundary condition between the massive body and the cloak, and the other one with a welded contact and no cloaking material, *cf.* Figure 8.

5.4 Mesh

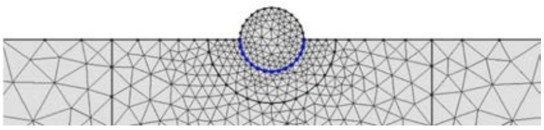

Figure 9: *The mesh*

The time step is a crucial part of the simulation procedure in COMSOL Multiphysics™. To illustrate this, values on ρ, μ, and ω have been used from the previous section. In order to obtain a suitable illustration the time increment has to be within the following tolerance

$$\Delta t \leq \frac{L}{C_s}$$

where L is the mesh element size, $C_s = \sqrt{\mu/\rho} = 1\,\text{m/s}$, is the shear wave speed and Δt is the time increment. In suitable units, the time increment will thus only depend on the element size: $\Delta t \leq L$.

5.5 Results

In our model we used the time-dependent solver in order to see the effects of Rayleigh wave propagation. The simulation ended at 10s with a time-step of 0.1s which was chosen to be less than the element size and sufficiently small, balancing accuracy against computing time.

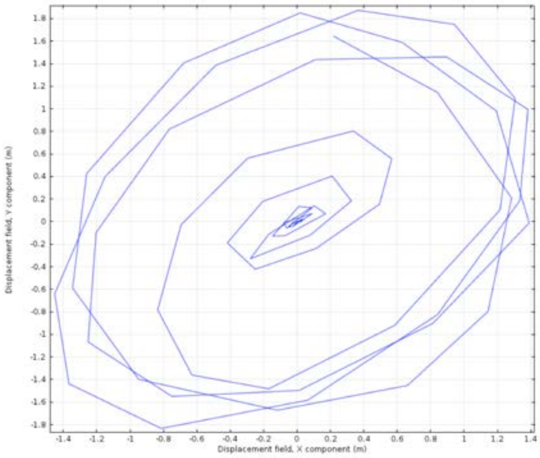

Figure 10: *Movement of pipeline in welded contact with homogenous material*

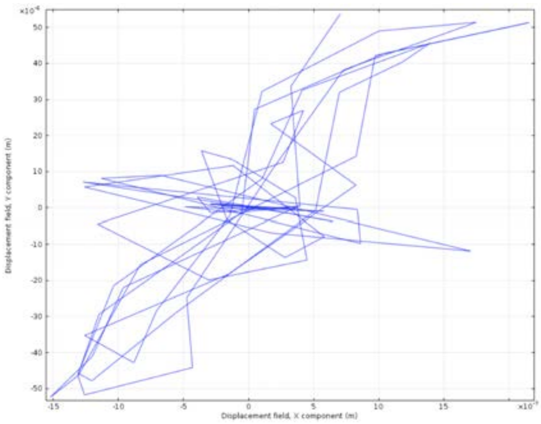

Figure 11: *Movement of pipeline rolling on a micropolar cloak*

Comparing the movements in the two cases, we see that the cloak in conjunction with rolling (or sliding) boundary conditions decreases the induced oscillation of the pipeline with many orders of magnitude! Already the use of rolling contact in fact brings a considerable decrease of the induced movement, but the details of this will be reported elsewhere.

6. Concluding Remarks

Summing up, we have found that elastodynamic cloaking from Rayleigh waves is theoretically possible, using a certain limiting type of such materials. However, this should come as no great surprise, as it has already been shown that cloaking against bulk waves is possible in that manner. In fact, the Rayleigh wave at a horizontal plane may be represented as a linear combination of such waves, albeit with imaginary wave numbers in the vertical direction. COMSOL Multiphysics™ proved to be useful, after some modification, to perform the modeling of this type of cloak.

We are presently also studying several alternative approaches to minimizing movements induced by elastic surface waves, and a comparison will be published elsewhere.

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