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Department of Mathematics

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Stability analysis of ALE-Methods for Advection-Diffusion problems

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Outline

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    1. Basic ideas
    2. Weak form of advection-diffusion problems on moving domains
3. Numerical calculations with COMSOL
    1. Weak form modeling
    2. Predefined ALE-mode
4. Conclusions and Outlook

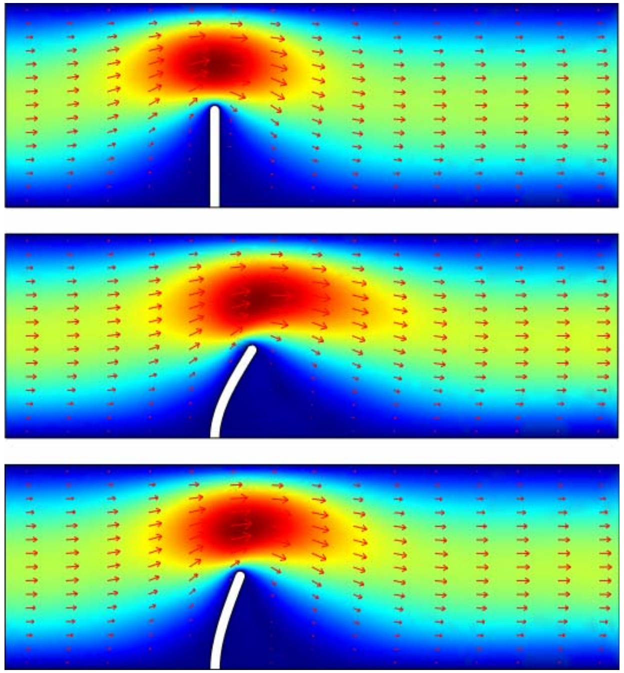
Motivation

Domain motion:

- ‘Level set’-method
- ALE-methods

Typical systems:

- Buildings in wind
- Haemodynamics



While for ALE-methods based on finite volume schemes certain conditions on the numerical schemes (*geometrical conversation laws*) have been shown [Far], similar results for FEM-schemes are still missing.

[Far] Farhat et al., *Comput. Meth. Appl. Mech. Engrg.*, **190**

Introduction

We consider a general parabolic advection-diffusion problem that can be written as:

$$\frac{\partial u}{\partial t}(x,t) + \mathcal{L}[u](x,t) = f(x,t) \qquad x \in \Omega \subset \mathbb{R}^n \text{ bounded}$$

$$t \in I = [t_0, t_E]$$

where $f \in L^2(\Omega \times I, \mathbb{R}^n)$

$\mathcal{L}[u] : I \times \mathbb{R}^n \to \mathbb{R}^n$ elliptic differential operator

Expect the domain to evolve in respect to time, where the domain displacement is given by an ALE-function a:

$$a : \Omega_0 \times I \to \mathbb{R}^n \qquad \alpha : \hat{\Omega} \to \Omega_0 \qquad \hat{\Omega} = \{(x,t), x \in \Omega_t, t \in I\}$$

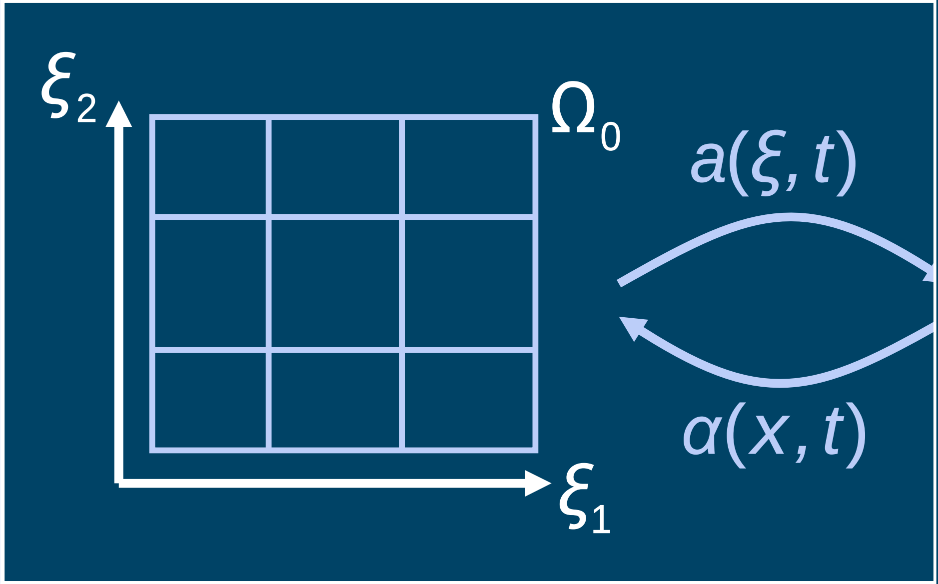
$$(\xi, t) \mapsto a(\xi, t) \qquad (x,t) \mapsto \alpha(x,t) \qquad \Omega_\tau = a(\Omega_0, \tau)$$

$$a(\alpha(\xi,t),t) = \xi$$

Basic ideas of ALE-methods

The basic idea of ALE-methods is to use different coordinate systems, a reference and a spatial system.

Calculation transformed to reference system!



Example:

$$\frac{\partial u}{\partial t}(x,t) + \mathcal{L}[u](x,t) = 0$$

weak formulation

$$\int_{\Omega_t} \psi(x,t)\cdot\frac{\partial u}{\partial t}(x,t)dx + \int_{\Omega_t} \psi(x,t)\cdot\mathcal{L}[u](x,t)dx = 0$$

domain transformation

$$\int_{\Omega_0} \psi(a(\xi,t))\cdot\frac{\partial u}{\partial t}(a(\xi,t),t)\cdot\det\frac{\partial a}{\partial \xi}(\xi,t)d\xi + \int_{\Omega_0} \psi(a(\xi,t))\cdot\mathcal{L}[u](a(\xi,t),t)\cdot\det\frac{\partial a}{\partial \xi}(\xi,t)d\xi = 0$$

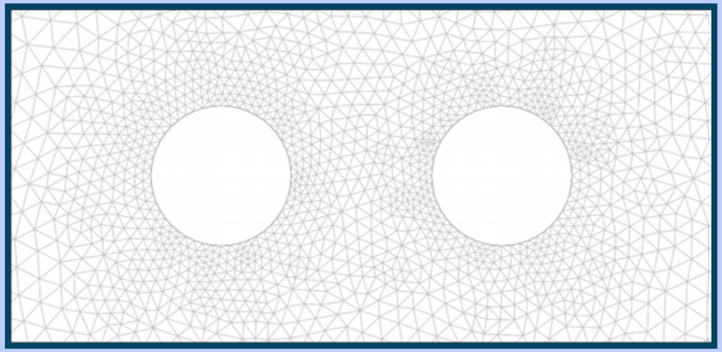
$a(\xi,t)$ continuous, invertible

$\alpha(\xi,t)$ continuous

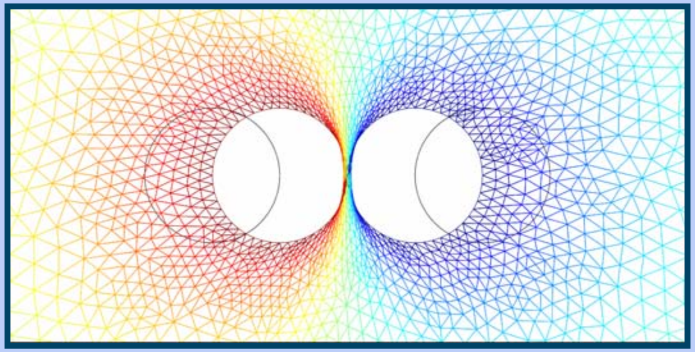
Limitations of ALE-methods

Limitations of ALE-methods:

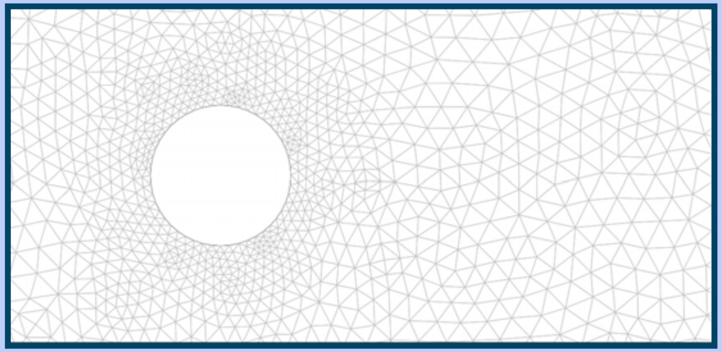
- Topological changes:



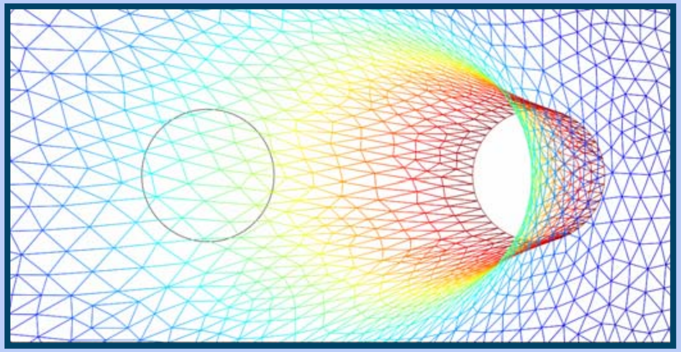
particles moving towards each other



- Very strong displacements:



particle moving too far in one direction



Weak form of ALE-equations

The original equation leads to the weak equation

$$\int_{\Omega_t} \psi(x,t) \cdot \frac{\partial u}{\partial t}(x,t)dx + \int_{\Omega_t} \psi(x,t) \cdot \mathcal{L}[u](x,t)dx = \int_{\Omega_t} \psi(x,t) \cdot f(x,t)dx$$

Recasting the equation in the reference system testfunctions can be chosen independent of t [Nob]:

$$\int_{\Omega_0} \hat{\psi}(\xi) \cdot \frac{\partial \hat{u}}{\partial t}(\xi,t) \cdot \det\left(\frac{\partial a}{\partial \xi}\right)(\xi,t)d\xi$$

$$+ \int_{\Omega_0} \hat{\psi}(\xi) \cdot \hat{\mathcal{L}}[u](\xi,t) \cdot \det\left(\frac{\partial a}{\partial \xi}\right)(\xi,t)d\xi$$

$$= \int_{\Omega_0} \hat{\psi}(\xi) \cdot \hat{f}(\xi,t) \cdot \det\left(\frac{\partial a}{\partial \xi}\right)(\xi,t)d\xi \qquad \text{where} \quad \hat{v}(\xi,t) = v(a(\xi,t),t)$$

[Nob] F. Nobile, *PhD Thesis*

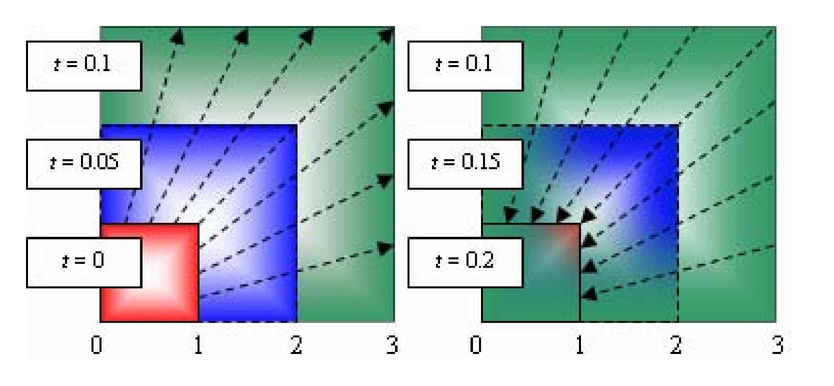
Model definition

Assume the following model system $\mathcal{L}[u](x,t) = -\Delta_x u(x,t)$ on the unit square $\Omega_0 = [0,1]\times[0,1]$

$$\frac{\partial u}{\partial t} - D\Delta_x u = f \qquad x \in \hat{\Omega}, t \in I$$

$$u(x,t) = 0 \qquad x \in \partial\Omega_t$$

$$u(\xi,0) = 16\cdot\xi_1(1-\xi_1)\cdot\xi_2(1-\xi_2) \qquad \xi \in \Omega_0$$



$$\alpha_i(x,t) = \frac{x_i}{2-\cos(10\pi t)} \qquad a_i(\xi,t) = \xi_i(2-\cos(10\pi t))$$

$$\hat{f}(\xi,t) = 40\pi\cos(5\pi t)\cdot\xi_1(1-\xi_1)\cdot\xi_2(1-\xi_2)$$
$$+\frac{32D\cdot(1+0.5\sin(5\pi t))}{(2-\cos(10\pi t))^2}(\xi_1(1-\xi_1)+\xi_2(1-\xi_2))$$
$$-\frac{160\pi\cdot(1+0.5\sin(5\pi t))\cdot\sin(10\pi t)}{2-\cos(10\pi t)}$$
$$\cdot\xi_1\xi_2(2-3\xi_1-3\xi_2+4\xi_1\xi_2)$$

Analytic solution is given by

$$\hat{u}(\xi,t) = 16\left(1+\frac{1}{2}\sin(5\pi t)\right)\cdot\xi_1(1-\xi_1)\cdot\xi_2(1-\xi_2)$$

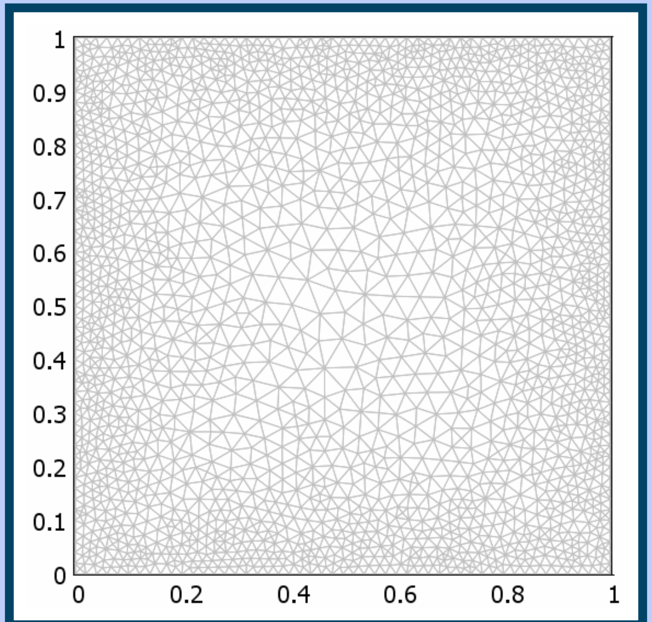
Weak formulation

The weak form in ALE-formulation is given by:

$$
\begin{aligned}
&(2-\cos(10\pi t))\int_{\Omega_0} \hat{\psi}(\xi)\frac{\partial \hat{u}}{\partial t}(\xi,t)d\xi \\
&\quad -\frac{D}{2-\cos(10\pi t)}\sum_i \int_{\Omega_0} \frac{\partial \hat{\psi}}{\partial \xi_i}(\xi)\cdot\frac{\partial \hat{u}}{\partial \xi_i}(\xi,t)d\xi \\
&\quad -10\pi\cdot\sin(10\pi t)\int_{\Omega_0} \hat{\psi}(\xi)\sum_i \xi_i \frac{\partial \hat{u}}{\partial \xi_i}(\xi,t)d\xi \\
&= (2-\cos(10\pi t))\int_{\Omega_0} \hat{\psi}(\xi)\cdot\hat{f}(\xi,t)d\xi
\end{aligned}
$$

The model was solved using implicit Euler scheme for $D = 0.01$ and $D = 1$ for

$\{\Delta t = 1/(20k),\ k = 1, 2, \ldots, 15\}$



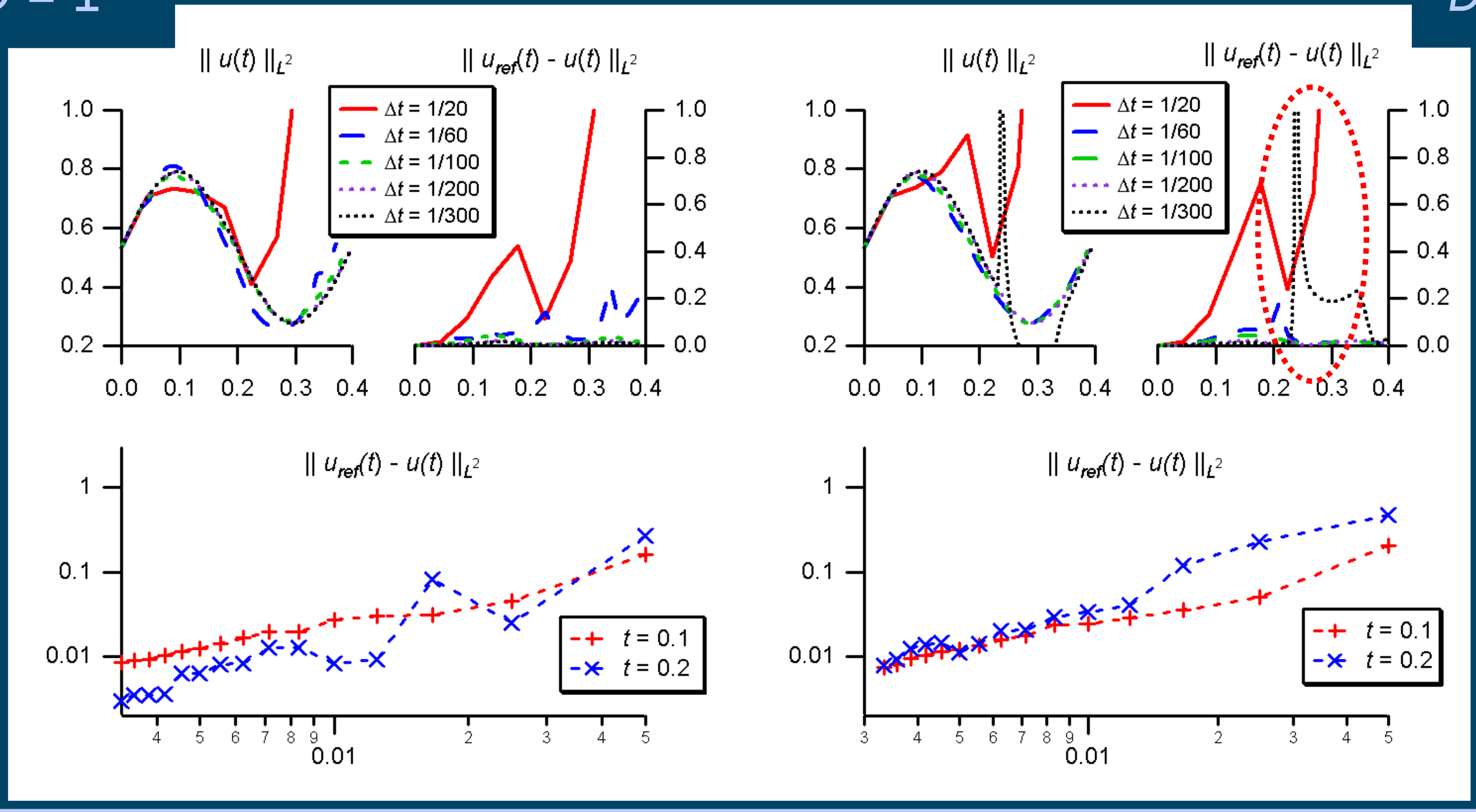
Max. elem. size at boundaries: 0.02

4098	elements
8397	degrees of freedom

Numerical results

Global error analysis

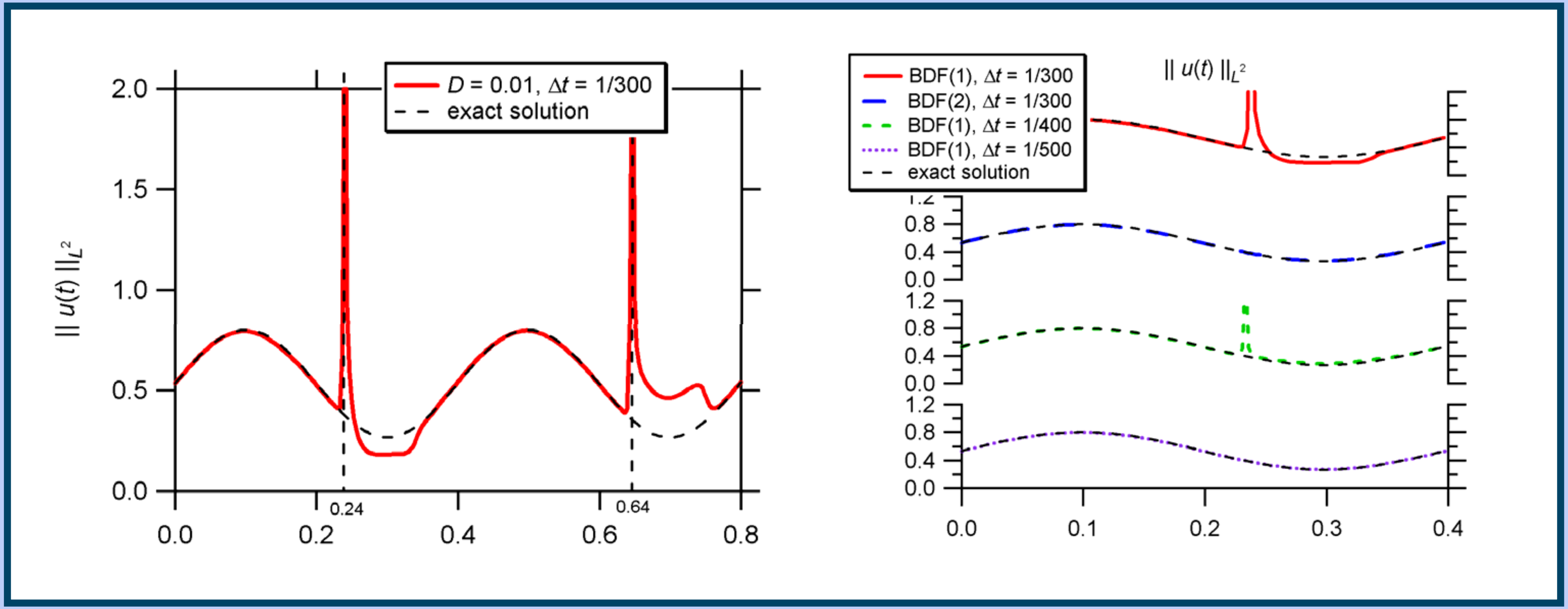
$D = 1$ $D = 0.01$



Strong deviations from the analytic solution can be found even for very fine timesteps !!

Numerical results

Global error analysis

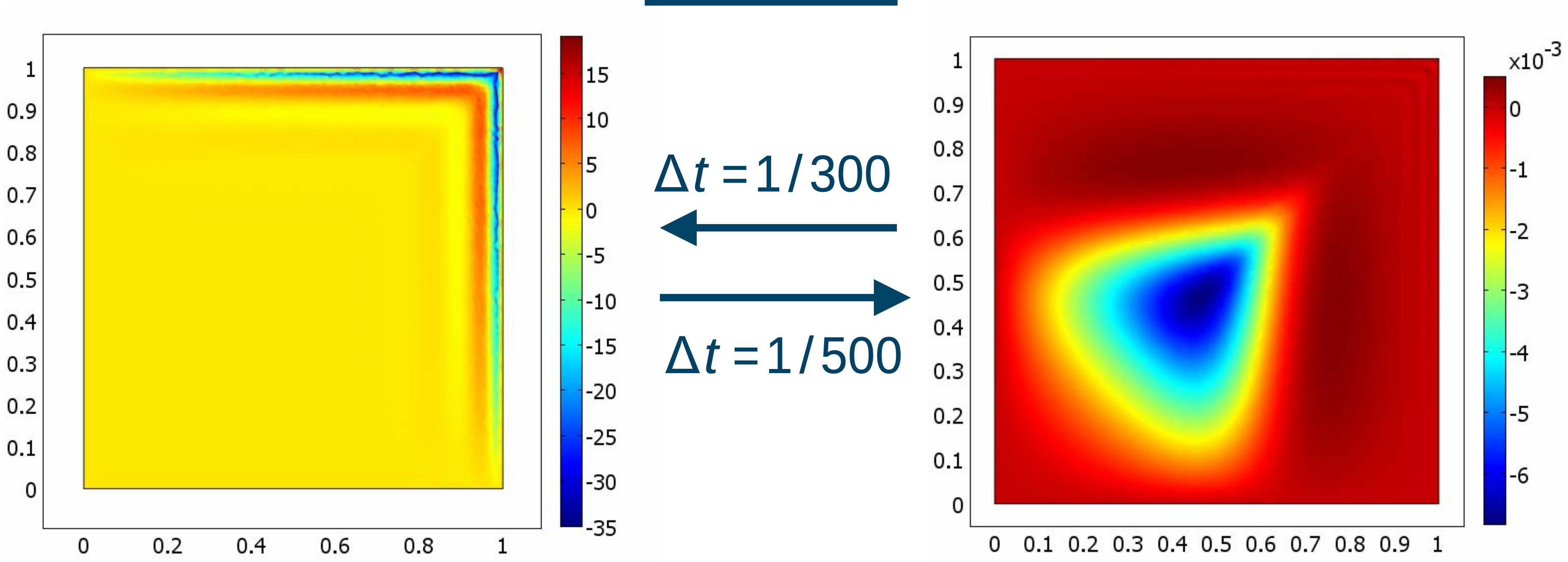


- peaks occur periodically
- even finer timesteps or higher BDF-order makes them disappear

Numerical results

Local error analysis

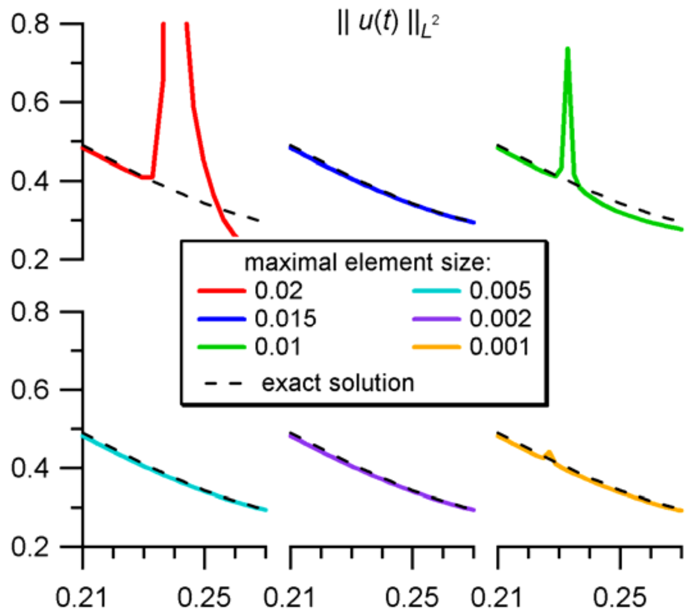
$D = 0.01$



- errors result from strong pointwise deviations close to the moving boundaries

Numerical results

Global error analysis



min. element size at upper and right boundary	Number of elements	Number of degrees of freedom
0.02	4098	8397
0.015	5194	10625
0.01	7507	15320
0.005	15140	30795
0.002	38770	78671
0.001	79252	160649

- even for very fine room discretization small deviations can be found

Predefined ALE-mode

Solving the equation with the predefined ALE-mode in COMSOL:

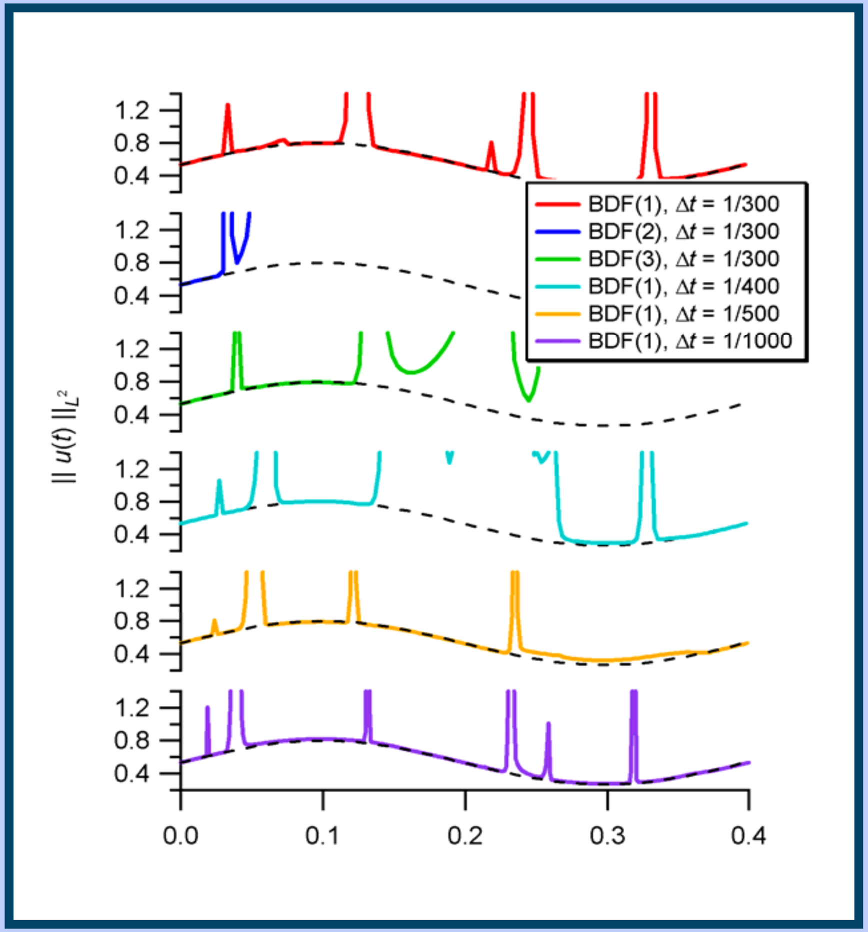
$$\int_{\Omega_t} \psi(x,t)\frac{\partial u}{\partial t}(x,t)dx$$

$$-D\sum_i \int_{\Omega_t} \frac{\partial \hat{\psi}}{\partial x_i}(x,t)\cdot\frac{\partial u}{\partial x_i}(x,t)dx$$

$$= \int_{\Omega_t} \psi(x,t)\cdot f(x,t)dx$$

where $$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial \alpha_j(x,t)}{\partial x_i}\frac{\partial}{\partial \xi_j}$$

Of course, equivalent on the continuous level, but **not** for the discrete schemes!



Conclusion & Outlook

Conclusion

- Large convection leads to instabilities in ALE-schemes
- Results depend strongly on discretization scheme: choose proper formulation in COMSOL Multiphysics.
- An increase of the BDF-order can lead to a “worse” result.

Outlook

- Find conditions for a numerical scheme with improved numerical stability
- Incorporate different stabilization methods used on fixed domains, e.g. Petrov-Galerkin-discretization