Application of an Electromagnetic Analogy in the Simulation of a Problem in Classical Hydrodynamics

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Abstract:

Consider the motion of an inviscid, constant-density fluid in an unbounded domain subject to a constant gravitational field and no other body forces. Suppose that the motion is: (a), two-dimensional *i.e.* there exists a plane $\mathcal{P}$ such that the fluid velocity field has spatial variations only in the directions parallel to $\mathcal{P}$; (b), localized, *i.e.* there is an origin O_L in $\mathcal{P}$ and a reference frame $\mathcal{F}_L$ relative to which the fluid at remote distances from O_L is at rest; and (c), *in the form of a traveling disturbance*, *i.e.* there is a second reference frame $\mathcal{F}_G$ and a point O_G fixed relative to $\mathcal{F}_G$ that translates rectilinearly with constant speed U relative to O_L such that the motion is stationary relative to $\mathcal{F}_G$. The present model simulates a motion that corresponds to the foregoing assumptions and whose results agree with an analytical solution of the same problem (see page 245 of Ref. 1), which I will call *Lamb's cylindrical dipole.*

Keywords: Unbounded domains, Hydrodynamic-Electrodynamic analogy, Equation-Based Modeling, Eigenvalue Solver Step, Eigenfunction

1. Stream function in two subdomains, one doubly-connected

Notation and partition into subdomains. Let $\{\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}\}$ be a right-handed set of constant unit vectors with $\hat{\mathbf{k}}$ perpendicular to-, and $\hat{\mathbf{\imath}}$ and $\hat{\mathbf{\jmath}}$ parallel to $\mathcal{P}$. Let $\hat{\mathbf{\imath}}$ point in the direction in which O_G translates relative to O_L. Let (x, y, z) and (ξ, η, ζ) be sets of cartesian coordinates belonging, respectively, to axis systems with origins at O_L and O_G and positive coordinate axes oriented in the directions of $\{\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}\}$ respectively, for both sets. Suppose that O_G and O_L coincide at time $t = 0$. Then

$$\xi = x - Ut \quad , \quad \eta = y \quad , \quad \zeta = z \, . \tag{1.1}$$

Here, and elsewhere, I employ Latin letters (or subscripts) to denote scalar components of vectors relative $\mathcal{F}_L$ and Greek letters (or subscripts) to denote scalar components of vectors relative $\mathcal{F}_G$. By the same token let $\mathbf{u}_L$ and $\mathbf{u}_G$ denote the fluid velocity vector relative to the frames $\mathcal{F}_L$ and $\mathcal{F}_G$, respectively. Similarly, let ∇_G and curl_G denote vector differential operators, in which the spatial derivatives are with respect to Greek letter positions coordinates and let ∇_L and curl_L denote vector differential operators in which the spatial derivatives are with respect to Latin ones. Since $\mathcal{F}_G$ translates rectilinearly relative to $\mathcal{F}_L$ in the direction of $\hat{\mathbf{\imath}}$ with constant speed U we have

$$\mathbf{u}_G = \mathbf{u}_L - U\hat{\mathbf{\imath}} \, . \tag{1.2}$$

A closed contour in a subdomain is *reducible* if one can, by a continuous deformation, shrink it to a point without touching the boundary of that subdomain. A subdomain is *simply-connected* if every closed contour in it is reducible. A *constant-density* fluid is one whose mass density ρ is both uniform in space and stationary in time.

Suppose that constant-density fluid fills an unbounded domain $\mathcal{D} = \mathcal{D}_b \cup \mathcal{D}_e$, in which $\mathcal{D}_b$ is a bounded simply-connected subdomain and $\mathcal{D}_e$ is its exterior. Then the only boundary of $\mathcal{D}_b$ is its outer edge $\partial\mathcal{D}_b$ and the only boundary of $\mathcal{D}_e$ is its inner edge $(\partial\mathcal{D}_e)_{\text{in}}$. In LAMB's example $\mathcal{D}_b$ is a circular disk of radius a centered on O_G. Thus the points in $\mathcal{D}_b$ and on $\partial\mathcal{D}_b$ satisfy the inequalities

$$\xi^2 + \eta^2 - a^2 \le 0 \quad \text{or} \quad (x - Ut)^2 + y^2 - a^2 \le 0 \tag{1.3}$$

and the points in $\mathcal{D}_e$ and on $(\partial\mathcal{D}_e)_{\text{in}}$ satisfy the inequalities

$$\xi^2 + \eta^2 - a^2 \ge 0 \quad \text{or} \quad (x - Ut)^2 + y^2 - a^2 \ge 0 \, . \tag{1.4}$$

The absence of time in $(1.3)_1$ and $(1.4)_1$ suggests that the use of the variables (ξ, η) as position coordinates wherever possible will be convenient and I will follow this approach in the sequel.

Consequences of the assumption of constant-density. Given an oriented contour with continually turning tangent $\hat{\mathbf{t}}$ (in the direction of the orientation) one can always construct a right-handed set of mutually-orthogonal unit vectors $\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{k}}\}$, in which $\hat{\mathbf{n}}$ is the local normal to the contour. If $\mathcal{C}$ is any reducible closed contour in either $\mathcal{D}_b$ or $\mathcal{D}_e$ the constant-density assumption implies that the net rate of transport of fluid volume (per unit depth) across $\mathcal{C}$ must vanish, *i.e.*

$$\oint_{\mathcal{C}} \mathbf{u}_G \bullet \hat{\mathbf{n}}\, d\sigma = 0\ , \tag{1.5}$$

in which (here and elsewhere) $d\sigma$ is the length of a typical small part of the contour in $\mathcal{F}_G$.

Now let $\mathcal{C}_{\rm nr}$ be a closed contour with clockwise orientation in $\mathcal{D}_e$ that encloses its inner boundary once . Then $\mathcal{C}_{\rm nr}$ is *not* reducible (which accounts for the choice of subscript). If one could show that

$$\oint_{\mathcal{C}_{\rm nr}} \mathbf{u}_G \bullet \hat{\mathbf{n}}\, d\sigma = 0 \tag{1.6}$$

for all choices of $\mathcal{C}_{\rm nr}$ then that fact, together with (1.5), would form the basis for an argument that the integral of $\mathbf{u}_G \bullet \hat{\mathbf{n}}\, d\sigma$ over an open contour between any two points, O and A, say, in either $\mathcal{D}_b$ or $\mathcal{D}_e$ is independent of the choice of path in between and, thence, an argument that $\mathbf{u}_G \bullet \hat{\mathbf{n}}\, d\sigma$ is an exact differential.

To justify (1.6) observe that the constant density assumption ensures that one can deform $\mathcal{C}_{\rm nr}$ into $(\partial\mathcal{D}_e)_{\rm in}$ without altering the value of the loop integral. After this deformation the result is the rate of transport of fluid volume (per unit depth) out of $\mathcal{D}_b$ into $\mathcal{D}_e$ across the interface between them. But the rate of transport of fluid volume (per unit depth) out of $\mathcal{D}_b$ is zero since $\mathcal{D}_b$ is filled with constant density fluid, which completes the justification of (1.6).

Having argued that $\mathbf{u}_G \bullet \hat{\mathbf{n}}\, d\sigma$ is an exact differential let $\mathbf{u}_G \bullet \hat{\mathbf{n}}\, d\sigma := d\psi_G$. A *streamline* is a contour that is tangent at all of its points, to the local velocity vector. If $\hat{\mathbf{t}}$ is the local unit tangent vector on a streamline then $\hat{\mathbf{t}} = \mathbf{u}_G/\|\mathbf{u}_G\|$ so $d\psi_G = \|\mathbf{u}_G\|\hat{\mathbf{t}} \bullet \hat{\mathbf{n}}\, d\sigma$, whose right member vanishes since $\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{k}}\}$ is an orthonormal system. Thus $d\psi_G = 0$ between any two neighboring points on a streamline. A streamline is therefore a contour of constant ψ_G. Following custom I will call ψ_G the *stream function* (relative to $\mathcal{F}_G$).

Moreover

$$d\psi_G = \mathbf{u}_G \bullet \hat{\mathbf{n}}\, d\sigma = \mathbf{u}_G \bullet(\hat{\mathbf{k}}\times\hat{\mathbf{t}})\, d\sigma = (\mathbf{u}_G\times\hat{\mathbf{k}})\bullet\hat{\mathbf{t}}\, d\sigma\ . \tag{1.7}$$

If one writes $\hat{\mathbf{t}}\, d\sigma = \hat{\mathbf{\imath}}\, d\xi + \hat{\mathbf{\jmath}}\, d\eta$ then the outermost equality in (1.7) becomes

$$d\psi_G = (\mathbf{u}_G \times \hat{\mathbf{k}})\bullet(\hat{\mathbf{\imath}}\, d\xi + \hat{\mathbf{\jmath}}\, d\eta)\ , \tag{1.8}$$

or, upon expansion,

$$\frac{\partial\psi_G}{\partial\xi}d\xi + \frac{\partial\psi_G}{\partial\eta}d\eta = u_\eta d\xi + (-u_\xi)d\eta\ . \tag{1.9}$$

If (1.9) is to hold for *all* combinations of the differentials $d\xi$ and $d\eta$ then (1.9) implies that

$$u_\xi = -\frac{\partial\psi_G}{\partial\eta} \quad , \quad u_\eta = \frac{\partial\psi_G}{\partial\xi}\ . \tag{1.10}$$

If one subtracts from (1.8) its counterpart in $\mathcal{F}_L$, [namely $d\psi_L = (\mathbf{u}_L \times \hat{\mathbf{k}})\bullet(\hat{\mathbf{\imath}}\, d\xi + \hat{\mathbf{\jmath}}\, d\eta)$], one gets

$$d(\psi_G - \psi_L) = [(\mathbf{u}_G - \mathbf{u}_L)\times\hat{\mathbf{k}}]\bullet(\hat{\mathbf{\imath}}\, d\xi + \hat{\mathbf{\jmath}}\, d\eta)\ .$$

But (1.2) implies that $\mathbf{u}_G - \mathbf{u}_L = -U\hat{\mathbf{\imath}}$, so

$$d(\psi_G - \psi_L) = [(-U\hat{\mathbf{\imath}})\times\hat{\mathbf{k}}]\bullet(\hat{\mathbf{\imath}}\, d\xi + \hat{\mathbf{\jmath}}\, d\eta) = U d\eta$$

or

$$d(\psi_G - \psi_L - U\eta) = 0\ . \tag{1.11}$$

The expression $\psi_G - \psi_L - U\eta$ must therefore equal a constant. The symmetries of this problem suggest that the ξ-axis should be a streamline in both $\mathcal{F}_G$ and $\mathcal{F}_L$. Consequently, the stream functions ψ_G and ψ_L are both uniformly constant on that axis and one may as well specify that they are both zero there. Indeed, I assume that both ψ_G and ψ_L are odd functions of η and periodic with respect to circuits about $\mathcal{C}_{\rm nr}$. The expression $\psi_G - \psi_L - U\eta$ must therefore vanish on the ξ-axis and so, being a constant, must vanish everywhere else. One thus arrives at the transformation rule

$$\psi_G = \psi_L + U\eta\ . \tag{1.12}$$

In the sequel I will seek a solution for the stream function that satisfies the homogeneous DIRICHLET condition

$$\psi_G = 0 \quad \text{on} \quad \partial\mathcal{D}_b\ . \tag{1.13}$$

The solution I seek will also be continuous across the interface $\mathcal{D}_b \cap \mathcal{D}_e$. Since $\partial\mathcal{D}_b$ and $(\partial\mathcal{D}_e)_{\rm in}$ are just the two sides of that interface the transformation rule (1.12) takes the the homogeneous DIRICHLET condition (1.13) for ψ_G to an inhomogeneous DIRICHLET condition for ψ_L, namely

$$\psi_L = -U\eta \quad \text{on} \quad (\partial\mathcal{D}_e)_{\text{in}} \, . \tag{1.14}$$

2. Boundary-value problem in the subdomain of irrotational motion

Formulation and analytic solution. The vorticity vector, $\boldsymbol{\omega}_L$, whose definition in $\mathcal{F}_L$ is

$$\boldsymbol{\omega}_L := \operatorname{curl}_L(\mathbf{u}_L) \, , \tag{2.1}$$

is a measure of the fluid spin, specifically twice the local angular rotation rate vector of a fluid particle. The motion is thus *rotational* if $\boldsymbol{\omega}_L \neq \mathbf{0}$ or *irrotational* otherwise. Since $\mathcal{F}_G$ does not spin relative to $\mathcal{F}_G$ the vorticities with respect to these frames must be equal, *i.e.* $\boldsymbol{\omega}_G = \boldsymbol{\omega}_L$ identically.

In the mean time the system (1.10) amounts to a list of the scalar components of the vector equation $\mathbf{u}_G = \operatorname{curl}_G(-\psi_G\hat{\mathbf{k}})$, whose counterpart in $\mathcal{F}_L$ is

$$\mathbf{u}_L = \operatorname{curl}_L(-\psi_L\hat{\mathbf{k}}) \, . \tag{2.2}$$

Under the present assumptions of two-dimensionality and for the present choice of coordinate axes one finds that two of the three scalar components of the vector equation $\boldsymbol{\omega}_G = \operatorname{curl}_G(\mathbf{u}_G)$ [*i.e.* the counterpart of (2.1) in $\mathcal{F}_G$] reduce to $0 = 0$. The remaining scalar component reads

$$\omega_\zeta = \frac{\partial u_\eta}{\partial \xi} - \frac{\partial u_\xi}{\partial \eta} \, . \tag{2.3}$$

Since the motion in $\mathcal{D}_e$ is irrotational by assumption ω_ζ must be uniformly zero there and (2.3) simplifies accordingly. If one substitutes the representation (1.10) into the simplified form of (2.3) one gets LAPLACE's equation for ψ_G, namely

$$0 = \frac{\partial^2 \psi_G}{\partial \xi^2} + \frac{\partial^2 \psi_G}{\partial \eta^2} \, . \tag{2.4}$$

Note, moreover, that the transformation rule (1.12) takes (2.4) to LAPLACE's equation for ψ_L, *i.e.*

$$0 = \frac{\partial^2 \psi_L}{\partial \xi^2} + \frac{\partial^2 \psi_L}{\partial \eta^2} \, . \tag{2.5}$$

If one rewrites (2.5) with respect to the polar coordinates (ϖ, ϑ) in accordance with the definitions

$$\xi = \varpi \cos\vartheta \quad , \quad \eta = \varpi \sin\vartheta \tag{2.6}$$

and multiplies the result by ϖ^2 one gets

$$\varpi \frac{\partial}{\partial \varpi}\left(\varpi \frac{\partial \psi_L}{\partial \varpi}\right) + \frac{\partial^2 \psi_L}{\partial \vartheta^2} = 0 \, . \tag{2.7}$$

The expressions $\ln(\varpi/a)$ and ϑ are both solutions of (2.7). As candidate solutions for ψ_G the so-called *line vortex* $\ln(\varpi/a)$ fails the conditions that it be an odd function of η and the so-called *line source* ϑ fails the condition that it be periodic with respect to circuits about $\mathcal{C}_{\text{nr}}$. One concludes that ψ_L has no contribution from either $\ln(\varpi/a)$ or ϑ.

Now $(\partial/\partial\eta)\ln(\varpi/a)$ is an odd function of η. Moreover, $(\partial/\partial\eta)\ln(\varpi/a) = \varpi^{-1}\sin\vartheta$ is a solution of (2.7), whose associated velocity as defined by (2.2), tends to zero as $\varpi \to \infty$. If one writes the DIRICHELT condition (1.14) in polar coordinates one gets

$$(\psi_L)_{\varpi=a} = -Ua\sin\theta \, . \tag{2.8}$$

A constant multiple of the so-called *line dipole* $\varpi^{-1}\sin\vartheta$ that satisfies (2.8) is

$$\psi_L = -U(a^2/\varpi)\sin\vartheta \, , \tag{2.9}$$

which must therefore be the analytic solution of the boundary-value problem for ψ_L in $\mathcal{D}_e$. This result agrees with LAMB's [see equation (68.4) on page 76 of Ref. 1]. Note next that equations $(2.6)_2$ and (2.9) take the transformation rule (1.12) to

$$\psi_G = U(\varpi - a^2/\varpi)\sin\vartheta \, , \tag{2.10}$$

which agrees with the corresponding result in LAMB [see equation $(68.10)_2$ on page 77 of Ref. 1].

COMSOL simulation. The main numerical challenge in the COMSOL simulation of the analytical solution (2.9) arises from the lack of an outer boundary of $\mathcal{D}_e$. The present approach to addressing this challenge begins with the observation that the pair of equations (2.1), (2.2) is analogous to a pair of equations that appear in the vector statement of AMPÈRE's law of electromagnetism. COMSOL's AC/DC module supports Infinite Elements and the present method for simulating the analytic solution (2.9) exploits this fact.

COMSOL's AC/DC module supports the Magnetic Fields (mf) Physics Interface. The first domain-level node under that Physics Interface is titled AMPÈRE's law. In the Settings window belonging to this node under Equation one finds a list of three equations the first two of which are

$$\operatorname{curl}\mathbf{H} = \mathbf{J}\ , \tag{2.11}$$

$$\mathbf{B} = \operatorname{curl}\mathbf{A}\ . \tag{2.12}$$

In the Hydrodynamic-Electromagnetic analogy I take the operator curl in the equations of Electromagnetism as a synonym for the operator curl_G in the corresponding Hydrodynamic problem. I also take the position coordinates (x, y) in the equations of Electromagnetism as synonyms for the coordinates (ξ, η) in the corresponding hydrodynamic problem. Further down in the same Settings window, under the heading Magnetic Field, the default Constitutive Relation is Relative Permeability, where one finds the equation

$$\mathbf{B} = \mu_0\mu_r\mathbf{H}\ , \tag{2.13}$$

in which constant values of μ_0 and μ_r are permissible as a special case. If one multiplies (2.11) by $\mu_0\mu_r$ and eliminates $\mathbf{H}$ from the left member by means of (2.13) one obtains

$$\operatorname{curl}\mathbf{B} = \mu_0\mu_r\mathbf{J}\ . \tag{2.14}$$

One may now construct a set of transformation formulas that take the the appropriate special case of the system (2.14), (2.12) to the system (2.1), (2.2), respectively. The appropriate special case of the magnetic potential vector $\mathbf{A}$ in the present two-dimensional problem is $\mathbf{A} = A_z\hat{\mathbf{k}}$, so (2.12) becomes

$$\mathbf{B} = \operatorname{curl}(A_z\hat{\mathbf{k}})\ . \tag{2.15}$$

Let B_s be a scalar constant having the same physical dimensions as the magnetic flux density $\mathbf{B}$. If one multiplies (2.14) and (2.15) by the common constant U/B_s these equations are equivalent, respectively, to

$$\frac{U}{B_s}\mu_0\mu_r\mathbf{J} = \operatorname{curl}\left(\frac{U}{B_s}\mathbf{B}\right)\ . \tag{2.16}$$

and

$$\frac{U}{B_s}\mathbf{B} = \operatorname{curl}\left(\frac{U}{B_s}A_z\hat{\mathbf{k}}\right)\ . \tag{2.17}$$

The system (2.1), (2.2) is now identical to the system (2.16), (2.17) under the transformation rules

$$\boldsymbol{\omega}_L = \frac{U}{B_s}\mu_0\mu_r\mathbf{J}\ ,\quad \mathbf{u}_L = \frac{U}{B_s}\mathbf{B}\ ,\quad -\psi_L = \frac{U}{B_s}A_z\ . \tag{2.18}$$

COMSOL's Magnetic Fields (mf) Physics Interface identifies the components of $\mathbf{A}$ as the dependent variables and allows the user to introduce a domain-level node titled External Current Density for specification of a given distribution of $\mathbf{J}$. There is an Application available within COMSOL via the path File > Application Libraries > AC/DC Module > Verification Examples > parallel_wires (Application ID: 131), which illustrates the procedure (including the use of an Infinite Element). In the present example of irrotational fluid motion $\boldsymbol{\omega}_L = \mathbf{0}$. According to $(2.18)_1$ the special case $\boldsymbol{\omega}_L = \mathbf{0}$ implies that $\mathbf{J} = \mathbf{0}$, so the electromagnetic analog of irrotational motion is current-free.

A COMSOL simulation of the analytic solution (2.9) [of the field equation (2.7)] must satisfy the far-field condition

$$\psi_L \to 0 \quad \text{as} \quad \varpi \to \infty\ . \tag{2.19}$$

The transformation rule $(2.18)_3$ takes (2.19) to

$$A_z \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty\ . \tag{2.20}$$

One may implement (2.20) in COMSOL by means of a Magnetic Insulation boundary condition at the outer edge of COMSOL's proxy for the Infinite Element. Equation $(2.18)_3$ also takes the DIRICHLET condition (1.14) to

$$A_z = B_s y \quad \text{on} \quad x^2 + y^2 = a^2\ , \tag{2.21}$$

which one may implement in COMSOL by means of a Magnetic Potential boundary condition.

In the simulations reported below I set B_s to $1\,[\mathrm{Wb/m^2}]$ and U to $1\,[\mathrm{m/s}]$ in the list of Global Parameters.

Under the heading Discretization I chose Cubic for the shape order. I also selected a User-controlled mesh. In the Size node under Mesh I chose Fluid Dynamics > Finer. Fig. 2.1 illustrates the result of COMSOL's calculation of the A_z–field in $\mathcal{D}_e$.

I introduced a COMSOL Variable Definition, whose scope is the annulus $a \leq r \leq 2a$, that expresses ψ_L in terms of the solution for A_z in accordance with the transformation rule $(2.18)_3$. Having

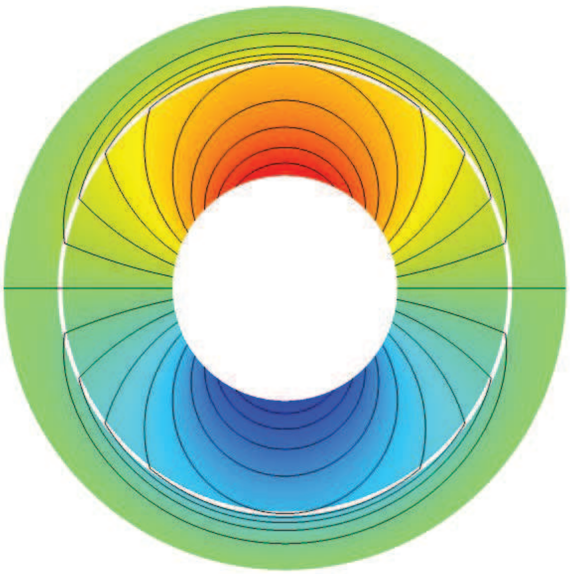

Fig. 2.1. Magnetic potential A_z exterior to disk of radius a centered at the origin governed by the current-free form of the field equations (2.12) & (2.14) and subject to the boundary conditions (2.20) and (2.21). The range (for both colors and contours) is $-1\,\mathrm{Wb/m^2}$(blue) $\leq A_z \leq 1\,\mathrm{Wb/m^2}$(red) and the increment between contours is 0.1 $\mathrm{Wb/m^2}$. The subdomain outside the large white circle is COMSOL's proxy for an Infinite Element.

ψ_L, I introduced a COMSOL Variable Definition, whose scope is again the annulus $a \leq r \leq 2a$, that defines ψ_G in accordance with (1.12). Fig. 2.2 nearby illustrates the result, which is equivalent to the corresponding figure in Lamb [see page 78 of Ref. 1]

3. Boundary-value problem in the subdomain of rotational motion

Lamb's analytic solution. A fluid is *barotropic* if the mass density ρ depends only on the pressure p. The case when ρ is a constant-valued function of

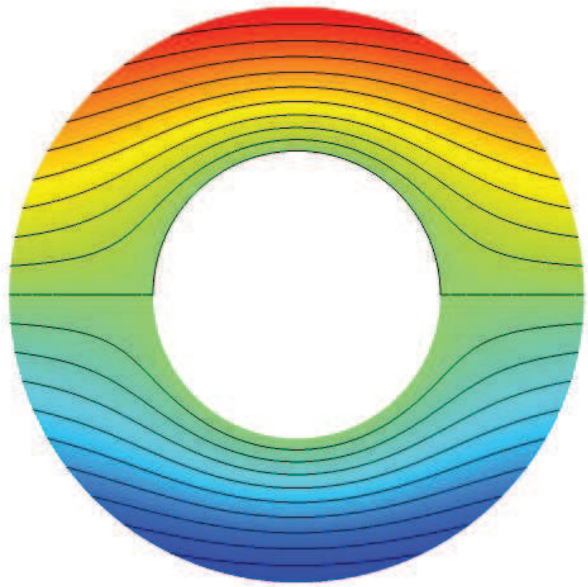

Fig. 2.2. COMSOL simulation of the stream function ψ_G in $\mathcal{D}_e$. The range is $-1.5\,Ua \leq \psi_G \leq 1.5\,Ua$ for for both colors and contours and the increment between contours is $0.15\,Ua$.

p is a special case of a barotropic fluid. If a fluid is inviscid and barotropic one may show that the vorticy $\boldsymbol{\omega}_G$ satisfies the equation

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\omega}_G}{\rho}\right)+\mathbf{u}_G \boldsymbol{\cdot} \nabla\left(\frac{\boldsymbol{\omega}_G}{\rho}\right) = \left(\frac{\boldsymbol{\omega}_G}{\rho}\right) \boldsymbol{\cdot} \nabla \mathbf{u}_G \; . \quad (3.1)$$

[see, *e.g.*, equation (146.4) on page 205 of Ref. 1 and its justification on pp 205–206]. The assumption of constant density enables one to simplify (3.1) by cancelling the common factor ρ. Furthermore, under the assumption of two-dimensionality, $\boldsymbol{\omega}_G = \omega_\zeta \hat{\mathbf{k}}$ and $\mathbf{u}_G$ does not vary in the direction of $\hat{\mathbf{k}}$, so the right member of (3.1) vanishes, and (3.1) simplifies again. In the mean time the time derivative term in (3.1) vanishes owing to the assumption that the motion in $\mathcal{F}_G$ is stationary, which yields a third simplification of (3.1). The one nontrivial component of (3.1) resulting from these three simplifications is

$$u_\xi \frac{\partial \omega_\zeta}{\partial \xi} + u_\eta \frac{\partial \omega_\zeta}{\partial \eta} = 0 \; . \quad (3.2)$$

Equations (1.10) take (3.2) to

$$-\frac{\partial \psi_G}{\partial \eta}\frac{\partial \omega_\zeta}{\partial \xi} + \frac{\partial \psi_G}{\partial \xi}\frac{\partial \omega_\zeta}{\partial \eta} := \frac{\partial(\psi_G, \omega_\zeta)}{\partial(\xi, \eta)} = 0 \; . \quad (3.3)$$

The vanishing of the Jacobian of the functions $(\xi, \eta) \mapsto \psi_G$ and $(\xi, \eta) \mapsto \omega_\zeta$ implies that there is a function $\psi \mapsto f(\psi_G)$ such that

$$\omega_\zeta = f(\psi_G) \ . \tag{3.4}$$

Now (2.3) takes (3.4) to $\partial u_\eta / \partial \xi - \partial u_\xi / \partial \eta = f(\psi_G)$, or, in view of (1.10)

$$\frac{\partial^2 \psi_G}{\partial \xi^2} + \frac{\partial^2 \psi_G}{\partial \eta^2} = f(\psi_G) \ . \tag{3.5}$$

LAMB considered the case $f(\psi_G) = -k^2 \psi_G$, in which k is a real constant with the dimensions of $(\text{length})^{-2}$(see page 245 of Ref. 1). Thus (3.5) becomes the two-dimensional HELMHOLTZ equation,

$$\frac{\partial^2 \psi_G}{\partial \xi^2} + \frac{\partial^2 \psi_G}{\partial \eta^2} + k^2 \psi_G = 0 \ , \tag{3.6}$$

A two-part representation of ψ_G that satisfies (2.4) in $\mathcal{D}_e$ and (3.6) in $\mathcal{D}_b$ will be physically meaningful only if satisfies the two continuity conditions

$$(\psi_G)_{\varpi = a^-} = (\psi_G)_{\varpi = a^+} \ , \tag{3.7}$$

$$(\partial \psi_G / \partial \varpi)_{\varpi = a^-} = (\partial \psi_G / \partial \varpi)_{\varpi = a^+} \ . \tag{3.8}$$

The right member of (3.7) vanishes by (1.13) so ψ_G in $\mathcal{D}_b$ is subject to the homogenous DIRICHLET condition

$$(\psi_G)_{\varpi = a^-} = 0 \ . \tag{3.9}$$

The trivial solution $\psi_G = 0$ in $\mathcal{D}_b$ satisfies both the homogeneous differential equation (3.6) and the homogeneous boundary condition (3.9) for all choices of the parameter k. Nontrivial solutions of the homogeneous boundary-value problem are possible only for special values of the parameter k, *i.e. eigenvalues.* A nontrivial solution of the homogeneous boundary-value problem belonging to a particular eigenvalue is an *eigenfunction.* Multiplication of an eigenfunction by a nonzero constant yields another eigenfunction. An eigenfunction is thus not unique. One can make it unique only by specifying a *normalization.*

Now (3.8) is an identity that is to hold for all choices of ϑ and would thus appear to specify more than a mere normalization. The condition (3.8) does, however, reduce to a normalization provided the left and right members depend upon ϑ through a common multiplicative factor. Equation (2.10) shows that the right member of (3.8) depends upon ϑ through the factor $\sin\vartheta$ so (3.8) will reduce to a normalization only if the left member does likewise.

If one rewrites (3.6) in polar coordinates in accordance with the the definitions (2.6) and multiplies the result by ϖ^2 one gets

$$\varpi \frac{\partial}{\partial \varpi}\left(\varpi \frac{\partial \psi_G}{\partial \varpi}\right) + \frac{\partial^2 \psi_G}{\partial \vartheta^2} + k^2 \varpi^2 \psi_G = 0 \ . \tag{3.10}$$

The definition of the nondimensional independent variable $w := k\varpi$ takes (3.10) to

$$w \frac{\partial}{\partial w}\left(w \frac{\partial \psi_G}{\partial w}\right) + \frac{\partial^2 \psi_G}{\partial \vartheta^2} + w^2 \psi_G = 0 \ . \tag{3.11}$$

If one substitutes trial solution proportional to $W(w)\sin\vartheta$ into (3.11) one finds, upon cancellation of the common factor $\sin\vartheta$ and rearrangement that W satisfies

$$w \frac{d}{dw}\left(w \frac{dW}{dw}\right) + (w^2 - 1)W = 0 \ , \tag{3.12}$$

which is the BESSEL equation of order 1. The BESSEL functions $J_1(w)$ and $Y_1(w)$ constitute a linearly independent pair of solutions of (3.12). A suitable solution for ψ_G in $\mathcal{D}_b$ is free of singularities. In the mean time the Bessel function $Y_1(w)$ is singular at the origin. If one recalls that $w = k\varpi$ one concludes that the only trial solution of (3.10) in $\mathcal{D}_b$ that is reconcilable with (3.8) is of the form

$$\psi_G = C J_1(k\varpi) \sin\vartheta \ , \tag{3.13}$$

in which C is a constant. Equations (2.10) and (3.13) take equations (3.7) and (3.8) to

$$J_1(ka) = 0 \ , \tag{3.14}$$

$$C J'(ka) k = 2U \ . \tag{3.15}$$

after division by the factors $C\sin\vartheta$ and $\sin\vartheta$, respectively. One can eliminate $J'(ka)$ in the left member of (3.15) by application of an identity that holds for BESSEL functions of integer order, namely $J_n'(w) = J_{n-1}(w) - (n/w)J_n(w)$ [*e.g.* equation (D) on page 360 of Ref. 2]. Substitution of $n = 1$ and $w = k\vartheta$ into this identity followed by application of (3.14) leads to the result $C = 2U/[kJ_0(ka)]$ [which appears with a sign error in equation (165.12) on page 245 of Ref. 1]. If one keeps five decimal places the smallest nonzero value of ka compatible with (3.14) is $ka = 3.83171$ [Ref. 3, page 409], which determines the eigenvalue k in the present problem.

COMSOL simulation COMSOL generated the results in Figs. 2.1 & 2.2 by means of a Stationary Study Step. I introduced a second Study Step via the path Study > Study Steps > Eigenfrequency > Eigenvalue. I also introduced second Physics Interface via the path Mathematics > Classical PDEs > Helmholtz equation (hzeq).

COMSOL reserves a special name for the eigenvalue, namely lambda. I set k to the equivalent of lambda/a. To ensure compatibility with the scale of the exterior motion and COMSOL's normalizaton I renormalized the eigenfunction so as to equate the integral of the tangential component of the fluid velocity over the semicircle $\varpi = a$, $0 \leq \vartheta \leq \pi$ as calculated from the solutions in the interior and exterior domains. Although the *integrals* with respect to arclength of the curves of $\mathbf{u}_G \bullet \hat{\mathbf{t}}/U$ on the inside and outside of the interface between $\mathcal{D}_u$ and $\mathcal{D}_p$ agree by construction I introduced no conditions that ensure that these two curves coincide *uniformly*. Be that as it may Fig. 3.1 shows that they do, thus confirming

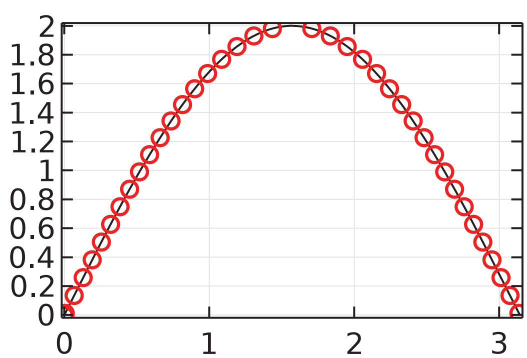

Fig. 3.1. COMSOL simulation of nondimensional tangential velocity $\mathbf{u}_G \bullet \hat{\mathbf{t}}/U$ versus polar angle ϑ on the upper sides of $(\partial\mathcal{D}_e)_{\text{in}}$ (black line) and $\partial\mathcal{D}_b$ (red circles). The polar angle ϑ (in radians) is zero at $(\xi, \eta) = (a, 0)$ and increases counter-clockwise.

the absence of slip across the interface. The shape of the curve corresponds, of course, to the common factor $\sin\vartheta$ in the left and right members of the continuity condition (3.8). Fig. 3.2 illustrates the distributions of vorticity (colors) and stream function (contours) for the bounded subregion of the flow plane.

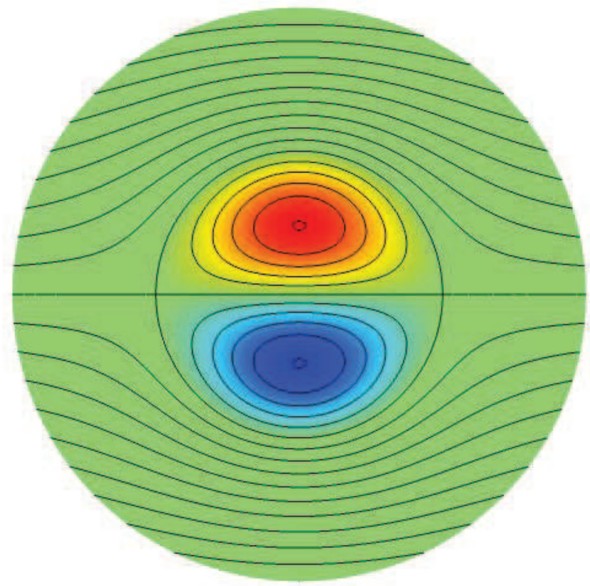

Fig. 3.2. COMSOL simulation of the vorticity ω_ζ (color) and stream function ψ_G (contours). The range for ψ_G is from -1.5 m^2/s to +1.5 m^2/s and the increment between contours is 0.15 m^2/s. The motionin the far field is from right to left. The value of ω_ζ within $\mathcal{D}_b$ is $-k^2\psi_G$, in which COMSOL's result for the eigenvalue ka is 3.8317.

The agreement between COMSOL's calculation of the eigenvalue (3.8317) and the handbook value (3.83171) is excellent. The corresponding figure for Lamb's analytical solution is graphically indistinguishable from Fig. 3.2.

7. References

1. Lamb, Sir Horace *Hydrodynamics*, Sixth edition, Cambridge University Press, 1932.

2. Whittaker, E.T. & Watson, G.N. *A Course of Modern Analysis.* Fourth edition, Cambridge University Press, 1927.

3. Abramowitz. M. & Stegun, I.A. *Handbook of Mathematical Functions.* National Bureau of Standards Applied Mathematics Series 55, 1964.